M.Sc. - Mathematics

I Semester End Examination - May 2022

Algebra – I

Course Code: MM101T
Time: 3 hours

QP Code: 11001
Total Marks: 70

Instructions: 1) **All** questions carry **equal** marks.
2) Answer **any five** full questions.

1. (a) Define a permutation on a set. Show that every permutation on a finite set is a product of disjoint cycles.
(b) Let ϕ be a homomorphism of G onto $\bar{G}$ with kernel K. Let $\bar{N}$ be a normal subgroup of $\bar{G}$ and $N = \{g \in G/\phi(g) \in \bar{N}\}$ then prove that $\frac{G}{N} \cong \frac{\bar{G}}{\bar{N}}$.
(c) Show that $T : G \to G$ defined by $T(x) = x^{-1}$ is an automorphism if and only if G is abelian.

(5+5+4)

2. (a) State and prove Orbit stabilizer theorem.
(b) For a finite group prove that $C_a = \frac{O(G)}{O(N(a))} = [G : N(a)]$.
(c) By using generator -relator form of S_3, verify the class equation of S_3, where S_3 is a symmetric group.

(5+5+4)

3. (a) If p is a prime number and $p|O(G)$ then prove that G has an element $a \neq e$ of order p.
(b) Show that any two subgroups of order p^n are conjugate to each other.

(7+7)

4. (a) Prove that every subgroup of a solvable group is solvable.
(b) Show that a normal subgroup N of G is maximal if and only if the quotient group G/N is simple.
(c) Prove that a group of order 36 is solvable but not simple.

(5+5+4)

5. (a) Define an integral domain. Prove that every field is an integral domain.
(b) Let R be a commutative ring with unity whose ideals are $\{0\}$ and R only. Prove that R is a field.
(c) Let R and R' be rings and ϕ is a homomorphism of R and R' with kernel U. Then show that $R' \cong R/U$.

(5+5+4)

6. (a) Let R be an integral domain with ideal P then prove that P is a prime ideal if and only if R/P is an integral domain.
   (b) Prove that an ideal of the ring of integers is maximal if and only if it is generated by some prime integer.
   (c)Define a prime ideal. Prove that in a commutative ring with unity a maximal ideal is always a prime ideal.

(4+5+5)

7. (a) Show that every field is a Euclidean ring.
   (b) If p is a prime of the form $4n + 1$ then show that $x^2 \equiv -1 \ (mod\ p)$ has a solution.
   (c) State and prove unique factorization theorem.

(4+5+5)

8. (a) Prove that $F[x]$ is a principal ideal ring where F is a field.
   (b) Define a primitive polynomial. Prove that the product of two primitive polynomials is primitive.
   (c) Verify that $f(x) = x^3 + x^2 - 2x - 1 \in Q[x]$ is irreducible polynomial by using Eisenstein criteria.

(5+5+4)

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